Life-Cycle Asset Allocation with Ambiguity Aversion and Learning *

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Abstract

Ambiguity and learning about the equity premium can simultaneously explain the low fraction of financial wealth allocated to stocks over the life cycle and the stock market participation puzzle. Individuals are ambiguous about the size of the equity premium and are averse to this ambiguity, resulting in lower stock allocations over the life cycle consistent with the data. As agents get older, they learn about the equity premium and increase their allocation to stocks. Furthermore, I find that ambiguity leads to underdiversification, home bias, lower Sharpe ratios, and higher savings. Similar results cannot be obtained by assuming higher risk aversion.

Keywords: Life-cycle portfolio choice, ambiguity aversion, learning, underdiversification
JEL classification: D14, D8, D91, G11

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I. Introduction

Key inputs of a life-cycle model, such as the equity risk premium, variance of stock returns, and labor income risk, are generally assumed to be known by the agent. Optimal portfolio allocations, consumption, and savings are calculated as if the agent takes these parameters as given and the resulting optimal policies are subsequently compared to the empirically observed life-cycle patterns. However, the predictions of most life-cycle models do not match well with some of the empirical findings. For instance, the overall low stock market participation rates are ill understood. Furthermore, the fraction of financial wealth allocated to stocks, conditional on participation in the stock market, appears difficult to align with the predictions from life-cycle models. I propose a standard life-cycle model, taking into account that agents are ambiguous about the equity risk premium and averse to this ambiguity (in contrast to the ambiguity-neutral approach). During their lifetime, individuals learn about the equity premium. With this parsimonious adjustment to the standard framework I can explain both the life-cycle pattern of participation in the stock market and the conditional fraction of financial wealth allocated to equity.

I assume that agents not only face risk, but also are uncertain about the true parameters describing this risk (Knight (1921)).[1] The ambiguity-neutral approach is a common way to deal with parameter uncertainty, where the decision maker treats the unknown parameters as random variables and combines his prior belief about the parameter with observed signals, which forms the predictive distribution. The decision maker then evaluates the expected utility with respect to this

[1]The difference between risk and uncertainty is that when agents face risk, they are able to attach probabilities to random events, while they do not know the probabilities when facing uncertainty. In the context of this paper, agents face risk because the returns on stocks are stochastic, but agents are also uncertain because they do not know expected stock returns.

predictive distribution. In this case, the agent is *ambiguous* but is *not ambiguity averse*. However, there is substantial evidence that agents are not neutral with respect to this parameter uncertainty (see, e.g., the classical paper by Ellsberg (1961), who argues that people are ambiguity averse by using an urn experiment). Therefore, I assume that agents are not ambiguity neutral, but ambiguity averse. Ambiguity about the equity risk premium is included, but no assumptions are made regarding the origin of this ambiguity. It can arise from, for instance, a lack of statistical evidence, a lack of theoretical evidence, a lack of investor sophistication, and so on. To illustrate, for example, statistical ambiguity, even when every agent possesses all the historical stock return data over the past 100 years and uses these to estimate the equity premium, the confidence interval will still be sizeable: $[4\% - 2 * 20\%/\sqrt{100} : 4\% + 2 * 20\%/\sqrt{100}] = [+0\% : +8\%]$.

A short note on terminology is in order. As Guidolin and Rinaldi (2013) point out, in the literature ambiguity and uncertainty are not always clearly defined. Throughout the paper I use the terms *uncertainty* and *ambiguity* interchangeably and I define *ambiguity/uncertainty* as a random event where the probabilities are not known (as opposed to a coin toss), but agents have a distribution of priors over the uncertain parameter.

I use maxmin preferences to model ambiguity aversion and individuals learn about the equity premium. Gilboa and Schmeidler (1989) propose that agents have maxmin preferences in a multiple-priors framework, which entails that agents evaluate policies by maximizing utility according to the worst case belief. This atemporal framework is generalized by Epstein and Schneider (2003) to a dynamic setup. I do not assume that agents learn about the equity risk premium in a rational manner; agents weigh realized stock returns during life with a prior belief about the equity risk premium, putting no weight on returns before birth. Malmendier and Nagel (2011) find that agents' "experienced return" has a larger influence on beliefs about the equity risk premium than stock return realizations before birth. I assume agents learn independently of stock market participation and I employ Bayes' rule as the updating rule for beliefs about the equity risk premium. Furthermore, agents have constant relative risk aversion preferences and their labor income is risky.

The contributions of this paper are fourfold. First, I find that ambiguity with respect to the equity risk premium can have a substantial effect on the optimal stock allocations. Stock market participation is substantially lower, as is the conditional allocation to equity. Both effects decline with age due to learning about the equity premium, since learning results in older agents being less ambiguous about the equity premium compared to younger agents. When disentangling the role of ambiguity and learning, I find that if agents are ambiguous but do not learn over time, they allocate on average a 5% lower fraction to stocks, and average participation rates are reduced by 5%.

Second, ambiguity-averse people hold more underdiversified portfolios when both ambiguous and non-ambiguous stocks are available. Ambiguity aversion leads to a tilt towards more familiar stocks, thereby providing a possible explanation for underdiversification and home-bias. This results in excessively risky holdings and lower Sharpe ratios for ambiguity-averse investors. As with the general stock allocation results, the effect of ambiguity aversion on underdiversification reduces with age.

Third, ambiguity about the equity premium influences the wealth levels of individuals, as well as their savings. The optimal wealth profile is lower for ambiguity-averse individuals, which reflects the lower investment returns due to lower optimal stock allocations. However, savings are higher when ambiguity about the equity premium is taken into account. This implies that keeping up wealth levels by saving extra to compensate for lower investment returns is quantitatively more important than lower savings induced by the relatively less attractive investment opportunities perceived by ambiguity-averse investors. This is the first paper, to my knowledge, to explore the impact of ambiguity on life-cycle savings choices and wealth levels.

Fourth, I present empirical evidence from the Survey of Consumer Finances consistent with the optimal allocations. Specifically, I find a close match at all ages to the empirical fraction allocated to stocks. On average over the life cycle the model predicts 45% of financial wealth allocated to stocks, similar to the empirical average of about 45%. A good fit is also found when examining participation in the stock market. In addition, the model with ambiguity generates a large degree of heterogeneity in fractions allocated to stocks, in line with the data. Hence, by extending the frequently used life-cycle model with ambiguity aversion and learning, I can simultaneously explain low stock market participation and the low conditional fraction of financial wealth allocated to stocks over the life cycle. Similar results cannot be obtained by assuming no ambiguity aversion and high risk aversion instead, because higher risk aversion actually increases participation levels due to higher precautionary savings. In addition, I find a negative relation between underdiversification and age in the data, consistent with the theoretical prediction from this life-cycle model. Finally, wealth-to-income ratios observed in the data are broadly in line with the theoretical levels.

In their seminal works, Merton (1969) and Samuelson (1969) find that agents should hold a constant fraction in risky assets over the life cycle in the absence of labor income and complete markets. More recent work by Benzoni et al. (2007), Cocco et al. (2005), Heaton and Lucas (2000), Polkovnichenko (2007), and Viceira (2001) examines the effect of (risky) labor income on the optimal portfolio choice. If human capital is riskless, young agents have a substantial investment in this "bond-like" asset and, as a result, invest a large fraction of their liquid wealth in risky assets. This is in contrast to the empirically observed low allocation to stocks, especially early in the life cycle. In contrast to other papers, I do not need to include several additional features in the model to explain low stock participation, such as participation costs (Vissing-Jørgenson (2002)), Epstein-Zin preferences, bequests, housing, cointegration between labor income and dividends, and minimum investment requirements. The intuitive modification with ambiguity aversion alone can explain the empirical evidence closely. Similar to this paper, Gomes and Michaelides (2005) try to match the empirically observed allocation to stocks, but they assume a bequest motive, fixed entry costs of 2.5% of income, preference heterogeneity, and Epstein-Zin preferences. The participation levels match closely, except after retirement; however, the predictions about the conditional allocation to equity differ about 40% from the empirically observed levels at younger ages. In contrast to aforementioned papers, I can match both low participation levels and the allocation to equity conditional on participation in the stock market, especially at young ages. Benzoni et al. (2007) assume cointegration between stock and labor markets and find a hump-shaped allocation to equity; however, the absolute differences from empirical levels are substantially larger than those in this paper.

One other paper includes ambiguity and learning about the parameters in a life-cycle framework and addresses similar questions as in this paper. Campanale (2011) assumes agents have maxmin preferences and are uncertain about the probability of a high stock return. The return on stocks can take on two values, high or low. Learning always occurs when agents invest in the stock market, but if they do not participate learning occurs with a probability below 100%. In contrast to my paper, this simplified stock return process prevents bringing this model to the data. Furthermore, I examine a broader set of choices, namely, diversification, savings, and wealth levels, and use a more parsimonious model.[2]

[2]Campanale (2011) includes a fixed annual stock market participation cost, a bequest motive, a minimum stock

Another related strand of literature explores the implications of ambiguity aversion on portfolio choice, where some of these papers also take learning into account. In contrast with the current paper, the literature described in this paragraph examines, to differing degrees, related questions, but from a non–life-cycle perspective. Within a mean-variance framework Garlappi et al. (2007) examine portfolio choice and Sharpe ratios with multiple ambiguous assets.[3] While the aforementioned paper focusses on the intensive margin, Cao et al. (2005) and Easley and O'Hara (2009) report that ambiguity aversion can help explain the extensive margin, limited stock market participation. Portfolio inertia can result from uncertainty aversion, which is documented in Dow et al. (1992). The influence of both ambiguity aversion and learning on portfolio choice in a continuous-time model is shown in Miao (2009) and Liu (2011). Furthermore, Boyle et al. (2012) find that a "flight to familiar assets" can arise if agents are less uncertain about some assets compared to others. Branger et al. (2013) show that strategies ignoring either ambiguity or learning can lead to large losses, where ambiguity is modelled about stock return predictability. Gollier (2011) explores the implications of ambiguity aversion both for portfolio choice and asset pricing. Specifically, this paper derives sufficient conditions for ambiguity aversion to lower the demand for uncertain assets and increase the equity premium. In a related paper Leippold et al. (2008) show that a model with both learning and ambiguity aversion can match the equity premium, interest rates, and the volatility of stock returns. Unfortunately, this overview of the literature on ambiguity aversion, learning, and portfolio choice is far from exhaustive, and two excellent and recent reviews are Epstein and Schneider (2010) and Guidolin and Rinaldi (2013).

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investment of 4% of average annual earnings in the economy (about $1,400), and a more complicated learning process.

[3]Similarly, Kan and Zhou (2007), and Tu and Zhou (2010) find that parameter uncertainty alters portfolio choices, while not assuming ambiguity aversion.

Several papers examine portfolio choice with uncertainty and learning while not assuming aversion to uncertainty. Gennotte (1986) shows that learning about the equity premium can lead to lower stock allocations by inducing a negative hedging demand. This learning generated hedge demand can be large, both when assuming mean excess returns are constant (Brennan (1998)) or time-varying (Xia (2001)). Pastor (2000) explores the impact on portfolio selection of the investors' prior degree of confidence in an asset pricing model, while Lundtofte (2008) models learning about the growth rate of the economy. Correlation between assets' expected returns is introduced in Cvitanic et al. (2006), which reduces uncertainty by inducing learning across assets. In contrast to several aforementioned papers, hedging demands in the life-cycle model with ambiguity aversion and learning are negligible due to slow learning since stock returns are highly volatile. The level of ambiguity (the size of the set of beliefs) at age 20 is reduced by less than 30% at age 100.

This paper differs from the previous literature in two important aspects. Namely, I examine ambiguity aversion and learning in the context of a life-cycle model, so I not only focus on the mean of stock market participation and conditional allocation to equity aggregated over all ages, but at all ages. Furthermore, I focus on a broad set of financial decisions of interest, such as the fraction allocated to stocks, stock market participation, underdiversification, and savings.

II. The model

I extend the standard life-cycle framework by including ambiguity aversion and learning. I use the most common model for ambiguity-averse preferences, namely, maxmin preferences.[4] Agents

[4]The robustness section uses α-maxmin preferences, an alternative way to model ambiguity aversion.

update their beliefs about the equity premium using Bayes' rule.

A. Ambiguity about the equity premium

I assume that agents are uncertain about the equity premium, but certain about the volatility of stock returns. Agents update beliefs according to realized stock market returns, which can be either actively or passively observed. The updating of beliefs follows from Bayes' rule, which is described in Section F. The initial prior belief about the equity premium is assumed to be normally distributed with mean λ^B and standard deviation σ^B. I further restrict the domain of equity premiums that the agents think are possible at time t to belong to the set $\Lambda_t = [\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$, where λ_t^B and σ_t^B are the mean and the standard deviation of the agents' belief at time t, respectively.[5] The latter assumption implies that every agent believes that the true equity premium lies in this interval with probability one. Garlappi et al. (2007) make a similar assumption when incorporating ambiguity by stating that the expected return of an asset lies within a specified confidence interval of its estimated value.

In what follows, I do not take a stand on the specific source of ambiguity about the equity risk premium. Uncertainty, for example, could stem from lack of statistical evidence, since stock market returns are very volatile and it is, thus, difficult to estimate the expected return.[6] Furthermore, ambiguity about the equity premium could also result from inconsistent theoretical evidence or lack of sophistication of investors.

[5]The set of priors satisfies the rectangularity condition (Epstein and Schneider (2003)).

[6]Even when using all the stock market returns from the past 100 years to estimate the equity premium, we would still end up with a large 95% confidence interval: $[4\% - 2 * 20\%/\sqrt{100}, 4\% + 2 * 20\%/\sqrt{100}] = [+0\%, +8\%]$.

B. Preferences

I consider a life-cycle investor of age $t = 1, ..., T$, where t is the adult age, T is the maximum age possible, and I denote by $K < T$ the retirement age. Individuals maximize utility over consumption and preferences are represented by a time-separable utility function over consumption. The agent's decision variables at time t are consumption, C_t, and the fraction of wealth invested in stocks, w_t. I assume investors' preferences are described by maxmin expected utility, which implies that agents maximize expected utility according to the belief that generates the lowest utility. Gilboa and Schmeidler (1989) axiomatize this behaviour in a static setting and Epstein and Schneider (2003) in a dynamic framework.

As described, the agent is uncertain about the equity premium and at every time t has a set of beliefs Λ_t. Moreover, at the beginning of period t the agent also observes the realization of wealth W_t (which depends on the realized past return) and income Y_t. The agent then selects optimal consumption and fraction of wealth invested in stocks so as to solve the following problem:

$$(1) \qquad V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t, C_t} \min_{\lambda \in \Lambda_t} \left\{ u(C_t) + \beta p_{t+1} \mathbb{E}_t^{\lambda} \left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B) \right] \right\}, \text{ with}$$

$$(2) \qquad u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma},$$

subject to all the constraints described in Section V.C, where β is the time preference discount factor. In addition, the probability of surviving to age $t+1$, conditional on having lived to period t, is indicated by p_{t+1}. The term $\mathbb{E}_t^{\lambda}$ denotes the conditional expectation computed using λ as the true equity premium. I assume a CRRA utility function, u, where γ is the risk aversion coefficient. It turns out that the minimum in the problem above is achieved when the agent uses $\lambda_t^B - 2\sigma_t^B$ as the equity premium. Note that the assumption that the agent's beliefs are limited to a bounded interval of possible equity premiums is not only intuitive, but also necessary since beliefs are normally distributed and, hence, the worst case belief would be unboundedly negative. In Section V.D, I assume α-maxmin preferences instead of maxmin preferences, in which case ambiguity aversion does not reduce to choosing the lowest admissible equity premium.

C. Constraints

The individual faces a number of constraints on consumption and investment decisions. First, I assume that the agent faces borrowing and short-sales constraints

$$w_t \geq 0 \text{ and } w_t \leq 1. \tag{3}$$

Second, I impose the following liquidity constraints on the investor:

$$C_t \leq W_t + Y_t, \tag{4}$$

which implies that the individual cannot borrow against future income to increase consumption today. The intertemporal budget constraint equals

$$W_{t+1} = (W_t - C_t + Y_t)R^P_{t+1}, \tag{5}$$

where R^P_{t+1} denotes the portfolio return

$$R^P_{t+1} = 1 + R^f + (R_{t+1} - R^f)w_t, \tag{6}$$

and R_{t+1} and R^f denote the stock return between time t and $t+1$ and the risk-free rate, respectively.

D. Financial market

I consider a financial market with a constant interest rate R^f and a stock with independent and identically distributed returns R_{t+1}. The stock returns, R_{t+1}, are normally distributed, with an annual mean equity return $R_f + \lambda^R$ and a standard deviation σ_R, where λ^R is the "true" equity risk premium. As described above, the agent does not observe λ^R and, at time t, believes that the equity premium lies in the set Λ_t, which changes over time due to learning. All the parameters employed are chosen in Section G.

E. Labor income process

I assume that labor income is uncertain and given by

$$Y_t = \exp(f_t + v_t + \epsilon_t) \text{ for } t < K, \tag{7}$$

where

$$v_t = v_{t-1} + u_t. \tag{8}$$

After the retirement age K, income is riskless and equals a fraction of the labor income at age 65 (the replacement rate). Labor income exhibits a hump-shaped profile over the life cycle that is accommodated by f_t, where f_t is a deterministic function of age. The error term consists of a transitory component and a permanent component. The term ϵ_t is a transitory shock and is distributed as $N(0, \sigma_\epsilon^2)$; u_t presents a permanent shock, where $u_t \sim N(0, \sigma_u^2)$. This representation follows Cocco et al. (2005) and I calibrate the labor income process according to their estimates. The function f_t is modeled by a third-order polynomial in age,

$$f_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2/10 + \alpha_3 t^3/100. \tag{9}$$

F. Learning and updating beliefs

Agents learn about the equity risk premium from birth throughout their lifetime and become less uncertain with age because they have received more information. The updating process for the set of priors follows Bayes' rule.[7] Before observing any signals, the set of priors is normally distributed, with mean λ^B and variance $(\sigma^B)^2$. I consider $t = 1$ to correspond to age 20, thus, an individual of age t has received $t + 19$ independent signals about λ^R, given by past returns $R_t = R^f + \lambda^R + \epsilon_t$, where ϵ_t is normally distributed with mean zero and a known variance σ_R^2. Signals are observed annually. The updated set of beliefs about λ^R are normally distributed, with mean λ_t^B and variance $(\sigma_t^B)^2$, where

$$\lambda_t^B = \lambda^B \underbrace{\frac{\frac{1}{(\sigma^B)^2}}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}}_{\text{weight mean prior}} + \frac{1}{t+19} \sum_{\tau=-19}^{t-1} (R_\tau - R^f) \underbrace{\frac{\frac{t+19}{\sigma_R^2}}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}}_{\text{weight returns}} \tag{10}$$

$$(\sigma_t^B)^2 = \frac{1}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}. \tag{11}$$

[7]Other updating rules for beliefs are explored by Epstein et al. (2010), Epstein and Schneider (2007), and Hanany and Klibanoff (2009).

The posterior mean λ_t^B is a precision-weighted average of the prior mean at birth and the average signal. Also, unlike λ_t^B, the posterior variance $(\sigma_t^B)^2$ does not depend on the specific realizations of the signals, only the number of signals. This variance, which measures the uncertainty/ambiguity about λ^R, decreases as the number of signals t increases (learning reduces uncertainty), that is, $(\sigma_{t+1}^B)^2 < (\sigma_t^B)^2$.

I assume agents update their beliefs irrespective of whether or not they participate in the stock market, e.g., since everyone receives similar information via the newspapers, television, and other media. People start with prior beliefs about the equity risk premium when born and update those beliefs according to realized returns from their birth year onward. The updating rule places no explicit weight on stock returns before birth and thus only takes into account realizations during one's lifetime. The priors at birth could be thought of as containing to some extent the realized stock returns before birth, but I do not assume that prior beliefs at birth are equal to the confidence interval from the stock return data available. I choose this specific starting age for updating beliefs instead of, for instance, adult age or long before birth, because Malmendier and Nagel (2011) find that stock returns experienced receive a much larger weight when beliefs are formed compared to stock returns before birth. In their baseline model, the authors use experienced returns dating back to the year of birth. In contrast to Malmendier and Nagel (2011), I apply equal weights to all experienced returns and update beliefs according to Bayes' rule.

Two additional underlying assumption are that (1) the level of ambiguity, σ^B, is the same for every person at birth, independent of birth year, and (2) the mean of the beliefs about the equity risk premium, λ^B, at birth is independent of birth year and hence independent of stock return realizations before birth. In regard to assumption (1), the reason I assume that the level of ambiguity (standard deviation of belief) about the equity risk premium is the same in 1970 and

2000 is that data going back more than, for instance, 70 years may, according to the agent, not be that relevant for estimating the equity premium today, due to, for example, structural changes (Pastor and Veronesi (2009)). Structural changes, induced by, say, technological innovations might permanently change the equity risk premium. Hence the amount of uncertainty does not disappear with time and is thus irrespective of the year in which the agent is born.

Assumption (2) is that the mean of the belief is the same for every person at birth and does not depend on birth year. Different priors at birth could generate additional cohort effects; however, I assume that the prior is independent of birth year, because agents incorporate realized stock returns during their life more heavily into beliefs than returns before birth (Malmendier and Nagel (2011)). In Section V, I explore the impact of starting updating from age 20 onward and different levels of initial ambiguity on the main findings. Furthermore, Section III disentangles the respective roles of ambiguity and learning.

G. Benchmark parameters for the life-cycle model with ambiguity

I set the risk aversion coefficient (γ) equal to five, which is the same as used by Benzoni et al. (2007) and Gomes and Michaelides (2005). Time ranges from $t = 1$ to time T, which corresponds to ages 20 and 100, respectively. Agents retire at time $K = 45$, corresponding to age 65. The survival probabilities are the current male survival probabilities in the United States, which are obtained from the Human Mortality Database.[8] I assume certain death at age 100.

Stock returns are normally distributed, and the true equity premium λ^R is 4% and the standard deviation σ_R is 16%, which is in accordance with historical stock returns. The risk-free rate is 2%;

[8]I refer to further information at the website at www.mortality.org.

hence the expected stock return is 6%. The mean of the priors about the equity premium at birth is equal to the true equity premium, $\lambda^B = 4\%$. The standard deviation of the beliefs at birth, σ^B, is 2%.

I use the parameters for the labor income process estimated by Cocco et al. (2005). The deterministic hump-shaped profile of income is generated by the parameters $\alpha_0 = 7.34$, $\alpha_1 = 0.1682$, $\alpha_2 = -0.0323$, and $\alpha_3 = 0.002$. The benchmark income level at age 20 is \$15,000, as in Cocco et al. (2005). The variance of the transitory shock to labor income, σ_u^2, is 7.38% and the variance of the permanent shock, σ_ϵ^2, is 1.06%. The replacement rate of labor income at age 65 is 68% of the wage at age 65. Income during retirement is riskless. These numbers are for a high school graduate as estimated by Cocco et al. (2005) and used as the benchmark parameters in their analysis.

H. The individual's optimization problem

The timing during one year is as follows: First, an individual receives labor or retirement income, after which the individual consumes. Subsequently the remaining wealth is invested. The optimization problem is solved via dynamic programming and I proceed backward to find the optimal investment and consumption strategy. In the last period the individual consumes all the remaining wealth; hence the individual's utility from terminal wealth is known.

The problem cannot be solved analytically, so I employ numerical techniques following Brandt et al. (2005) and Carroll (2006) with several extensions by Koijen et al. (2010). Brandt et al. (2005) adopt a simulation-based method that can deal with many exogenous state variables. In this model, the mean of the beliefs about the equity premium, λ_t^B, and income, Y_t, are the relevant exogenous state variables. Wealth acts as an endogenous state variable. For this reason, following

Carroll (2006), I specify a grid for wealth *after* income and consumption. As a result, I do not need numerical root finding to obtain the optimal consumption decision. The details of the numerical method I use to solve the life-cycle problem with maxmin preferences are described in Appendix A.

I. Data

When comparing the predicted stock allocation to the data, I use the 2010 Survey of Consumer Finances, which is the most comprehensive dataset on U.S. household assets and liabilities. High-income households are over-sampled to obtain a sufficient number of wealthy households in the study. I employ a measure for financial wealth and stock investment according to the method suggested by the Survey of Consumer Finances. The same measures are used by Gomes and Michaelides (2005), as is the measure for wealth-to-income ratio that I employ. Financial wealth consists of both retirement and non-retirement wealth and stock investment is calculated as the sum of direct investment in stock and stock mutual funds, as well as stock investments of pension wealth. More details on the data from the Survey of Consumer Finances can be found in Appendix C.

III. Effect of ambiguity aversion on optimal allocations

I determine the optimal life-cycle choices using simulations and analyze the importance of ambiguity about the equity premium on optimal allocations.

A. Effect of ambiguity aversion and learning on optimal portfolio choice

The optimal fraction allocated to stocks, conditional on participation in the stock market, is plotted in Figure 1a. Comparing optimal allocations including ambiguity and learning (solid line) to those with no ambiguity (dashed line) shows that the allocation to stocks when agents are ambiguity averse is much lower. The average fraction allocated to stocks over the life cycle is around 45% instead of 80%. Furthermore, the effect of ambiguity on stock allocations dampens with age, as agents learn about the equity premium. This can be seen by the narrowing of the gap between the stock allocation with ambiguity and learning (solid line) and no ambiguity (dashed line).

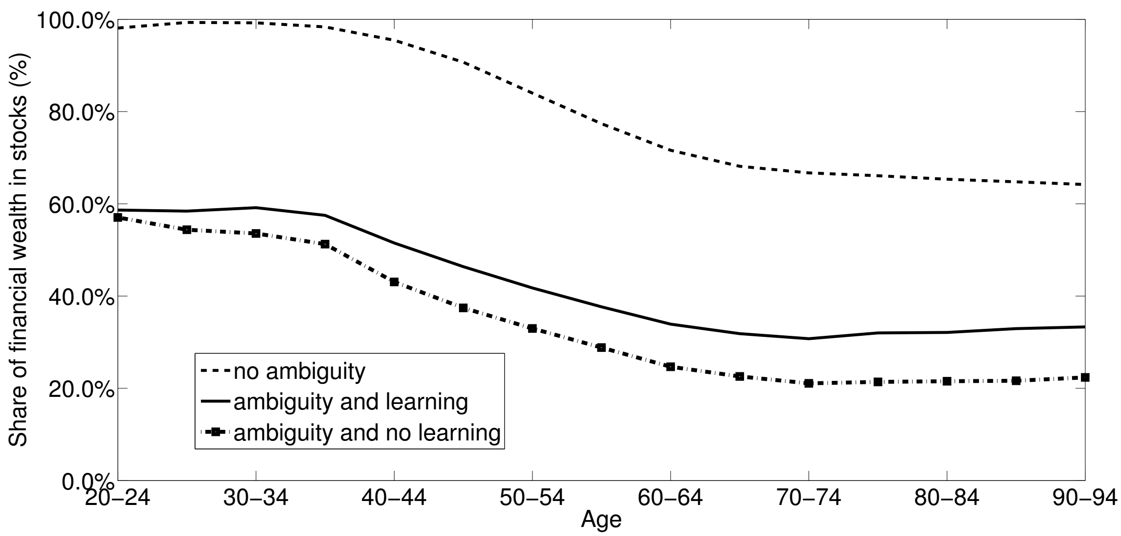
In the baseline model I assume that agents learn every year, updating their beliefs about the equity premium depending on the specific realisation. To disentangle the effect of of learning and ambiguity aversion, I assume that agents keep their beliefs at age 20 fixed during the remainder of their life. The dashed-dotted line presents the impact of ambiguity in the absence of learning. The fraction allocated to stocks conditional on participation is lower when agents do not learn with age. This is intuitive as ambiguity does not lessen over time due to learning, since the standard deviation of the belief about the equity premium, σ_t^B, does not shrink with every additional stock return realisation experienced. The difference in the fraction allocated to stocks with and without learning is 5% on average over the life cycle, and the difference in participation levels is also about 5%. This reflects that learning is slow, since stock returns are highly volatile.[9]

Focusing on the analysis without ambiguity, I find that if agents are fully certain about the values of all the parameters in the model, they allocate 100% of financial wealth to stocks before age 40. Similar results are found by Cocco et al. (2005). The reason for this high fraction is that

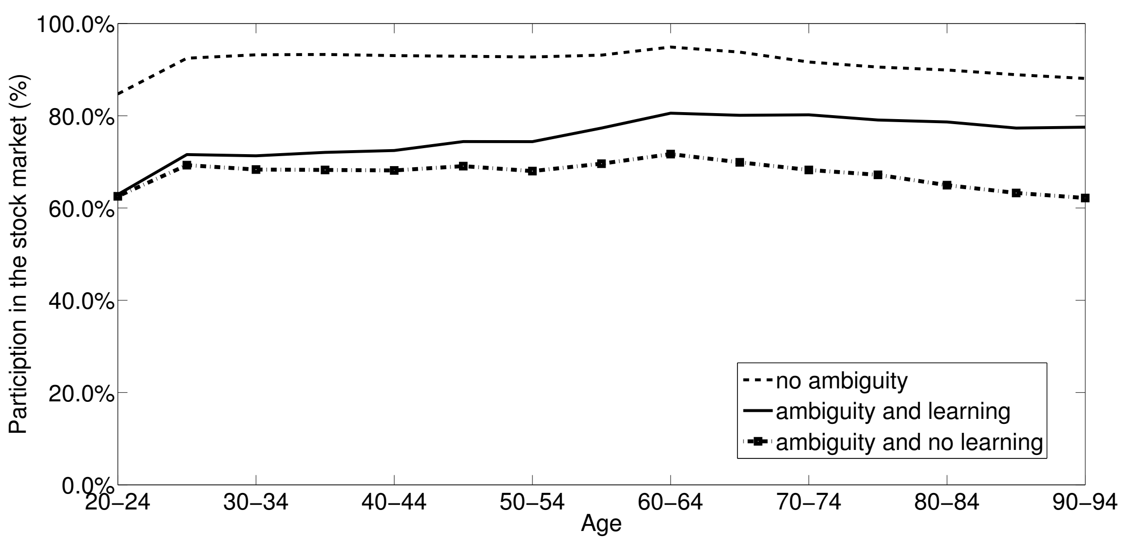
[9] The set of beliefs about the equity premium at age 20 is [0.56%,7.44%] and if agents learn this set at age 100 is [1.54%,6.46%].

Figure 1: Optimal fraction allocated to stocks and optimal participation in the stock market
These figures show the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) agents ambiguous about the equity risk premium, averse to this ambiguity, and who learn about the equity premium, (2) agents not ambiguous about the equity premium, and (3) agents ambiguous about the equity risk premium, averse to this ambiguity, and who do not learn about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation



young agents have only a small amount of financial wealth compared to a high level of human capital. Since human capital is like an implicit investment in a riskless asset, agents allocate their entire financial wealth to equity. Between ages 40 and 65, the conditional allocation to risky assets decreases. At these ages, (retirement) savings are high while at the same time the net present value of labor income decreases; hence the fraction of financial wealth to human capital increases. This results in a decline of the relative allocation to the riskless asset human capital and, consequently, the optimal fraction of financial wealth invested in stocks decreases to maintain a similar risk profile.

Figure 1b displays optimal participation levels in the stock market. The effect of ambiguity is substantial and participation levels drop by 20% on average over the life cycle. When agents are not ambiguous about the equity premium, the participation levels in the stock market are high. Since labor income is not correlated with returns on the stock market, it is optimal for all agents, even those with low financial wealth, to allocate at least a small fraction of financial wealth to stocks. The reason for below 100% participation is that I assume that agents with financial wealth less than $100 do not invest in stocks. Taking these agents with near-zero wealth into account would distort the subsequent comparison of the model predictions to the data, since in reality people with less than $100 of wealth would not invest, due to participation costs and minimum balance requirements.[10] As before, the impact of ambiguity aversion decreases with age, since the ambiguity about the equity risk premium declines as agents learn by observing the realized stock returns. In a non–life-cycle framework, Cao et al. (2005) and Easley and O'Hara (2009) confirm that ambiguity aversion can limit participation levels.

[10]In addition, the simulation inaccuracy of optimal stock allocations is higher for these low wealth levels, since the difference in the utility of the agent investing 100% or 0% in stocks is negligible.

In Section V.C I show the sensitivity of the results to a different level of ambiguity at birth. In the baseline I choose a conservative measure of ambiguity, the standard deviation of the belief about the equity premium at birth is 2%. I document that when assuming 3% the effect on participation is much larger, while the effect on the fraction allocated to stocks remains the same.

Vissing-Jørgenson (2002) examines the implications of fixed participation costs on optimal participation levels and finds that it can explain why less wealthy households do not participate, but not the low participation levels of the wealthy. I find that ambiguity about the equity risk premium can provide an explanation for the low participation levels of wealthy individuals as well. Furthermore, ambiguity about the equity premium can simultaneously explain the low participation levels, as well as the low conditional fraction allocated to stocks, while fixed participation costs only impact participation levels.

The previous paragraphs explore the optimal allocations for agents who are ambiguous *and* averse to this ambiguity. In contrast, in the more standard ambiguity-neutral framework, agents are only uncertain about the parameters, but not averse with respect to this uncertainty. When this is the case, the optimal allocations hardly change. In the benchmark model, the agents' beliefs about the equity risk premium are normally distributed, with a mean of 4% and a standard deviation equal to 2% at birth. If agents are ambiguity neutral, their behaviour is induced by the so-called predictive distribution. The standard deviation for the compound distribution of the volatility of the return on equity, σ_R and the volatility of the belief, σ_t^B, can be reduced to the predictive volatility $\sqrt{\sigma_R^2 + (\sigma_t^B)^2}$. For the benchmark parameters, this results in a standard deviation of 16.1% (note that σ_R is 16%). Hence uncertainty about the equity risk premium will have (almost) no effect on optimal portfolio choices when uncertainty neutrality is assumed.

B. Effect of ambiguity aversion on underdiversification and Sharpe ratios

The previous section shows that ambiguity aversion can provide a theoretical foundation for lower stock holdings, both at the extensive and intensive margin. In this section, I explore whether ambiguity aversion has implications beyond allocation to stocks as an asset class, focussing on particular categories of stocks. Specifically, I explore the potential implications of ambiguity aversion on underdiversification and Sharpe ratios. A large empirical literature shows that investors tend to hold underdiversified portfolios and exhibit a home bias. Foreign stocks tend to be more ambiguous compared to domestic stocks for many investors, which can potentially lead to home bias. More generally, agents will perceive various stocks to have different levels of ambiguity. This familiarity towards certain stocks compared to others may arise from, for instance, geographical proximity or employment relationships. I test whether ambiguity aversion can potentially lead to underdiversification, either by not investing in this ambiguous asset, or by allocating a lower fraction to an ambiguous asset compared to a non-ambiguous asset. Furthermore, I show the evolution of underdiversification over the life cycle.

I assess the impact of ambiguity aversion on underdiversification by adding to the baseline asset menu a risky, but not ambiguous asset. This asset has the same "true" equity premium, λ^R, and standard deviation ,σ_R, as the ambiguous asset and I assume zero correlation between the two assets. Hence the agent can choose to invest in a risky, non-ambiguous asset, a risky, ambiguous asset, and a riskless asset. Details of the model are in Appendix B. I focus on three measures of diversification. One is the number of risky assets held, conditional on participating in the stock market. In the model there are two stocks available hence two generates the highest level of diversification achievable. Furthermore, I explore a second measure for diversification to account for the fact that the portfolio can still be underdiversified even if an individual holds both assets, namely by holding a non-equal fraction in both stocks. Therefore, the second measure of diversification is the fraction of the total stock portfolio invested in the ambiguous stock, conditional on investing in both stocks. Furthermore, following Calvet et al. (2009), I use a third measure for underdiversification, the relative Sharpe ratio loss, which allows assessing the costs of underdiversification in terms of a lower Sharpe ratio.

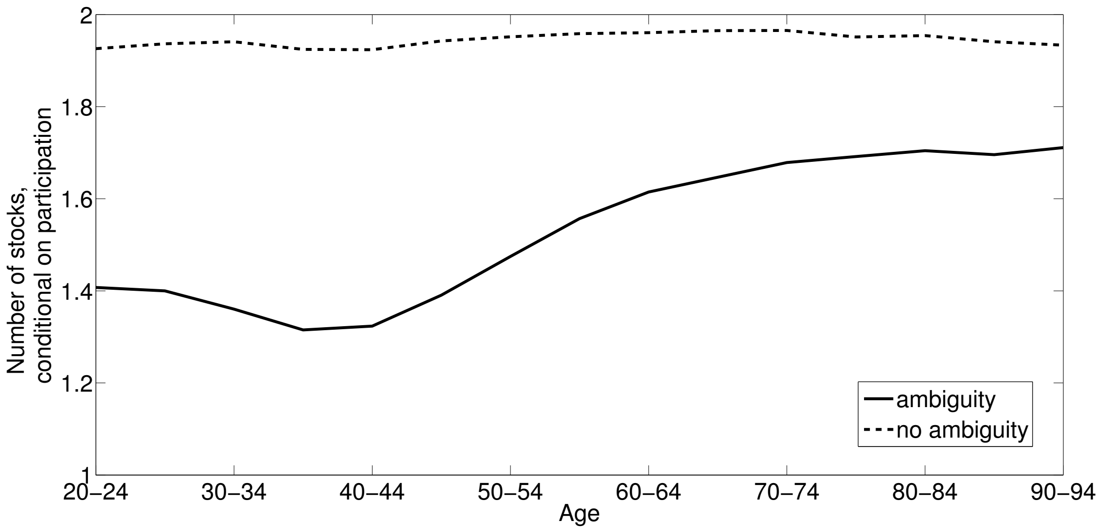
Figure 2 shows the number of stocks over the life cycle, both for the case with ambiguity and without ambiguity. An individual optimally invests in both stocks when not ambiguity averse, thereby achieving a lower level of overall volatility of the portfolio.[11] If the second asset is ambiguous and a person is ambiguity averse, then the optimal number of stocks is lower, on average over the life cycle about 1.5 stocks. The ambiguous asset is less attractive due to the aversion to ambiguity, and the benefit of a lower portfolio volatility due to diversification is not enough to compensate. Furthermore, I find that underdiversification decreases with age. Agents learn about the mean excess return of the ambiguous stock, therefore increasing their investment and participation in this ambiguous stock. In Section IV.C, I test whether this life-cycle pattern is consistent with the data.

When focussing on the second measure, I find that ambiguity aversion leads to a lower fraction invested in the ambiguous asset compared to the non-ambiguous asset, conditional on participating in both assets. This result is presented in Table 1. An individual invests on average only 31% of the total stock portfolio in the ambiguous stock, conditional on investing in both stocks. This tilt towards the non-ambiguous stock leads to excessively risky stock portfolios. In a non–life-cycle context, Boyle et al. (2012) and Guidolin and Liu (2015) also find that ambiguity aversion lowers

[11]The number of assets invested is slightly below two due to simulation error.

Figure 2: Diversification over the life cycle: number of stocks in stock portfolio

The figure displays the number of stocks invested in, conditional on stock market participation. The dashed line shows the diversification measure when an agent does not face ambiguity. In that case two risky stocks are available with the same mean and volatility, and zero correlation. The solid line shows the diversification measure when an agent does face ambiguity. In this case also two stocks are available: the first stock is ambiguous and risky, and the second stock is risky, but not ambiguous. Both stocks have the same mean and volatility, and correlation is zero. A riskless asset is available both in the ambiguity and no-ambiguity setup. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.



portfolio diversification. Confirming the theoretical result, Dimmock et al. (2015a) employ survey data and find evidence that ambiguity-averse investors hold less diversified portfolios, compared to non–ambiguity-averse investors.

Furthermore, underdiversification will lead to lower Sharpe ratios for ambiguity-averse agents, compared to non–ambiguity-averse agents, which is shown in Table 1. The Sharpe ratio is lowered because in many cases agents invest only in the risky asset, and not in the ambiguous asset. In addition, when investing in the ambiguous asset, they invest a smaller fraction in this asset compared to the risky asset, lowering the Sharpe ratio further. An ambiguity-averse agent has on average over his lifetime a Sharpe ratio of 0.27, compared to 0.33 if an agent does not face ambiguity. Hence the third measure for underdiversification, the relative Sharpe ratio loss, is 18%. In contrast, Garlappi

Table 1: Diversification measure and Sharpe ratio

The diversification measure is the fraction of the total stock portfolio invested in the ambiguous stock, conditional on investing in both the ambiguous and non-ambiguous stock. The Sharpe ratio is the average realized excess return divided by the standard deviation of excess returns. Subsequently the measure is averaged over the life cycle. The column "no ambiguity" shows the diversification measure when an agent does not face ambiguity. In that case two risky stocks are available with the same mean and volatility, and zero correlation. The column "ambiguity" shows the diversification measure when an agent does face ambiguity. In this case also two stocks are available: the first stock is ambiguous and risky, and the second stock is risky, but not ambiguous. Both stocks have the same mean and volatility, and correlation is zero. A riskless asset is available both in the ambiguity and no-ambiguity setup. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

	no ambiguity	ambiguity
Diversification measure	0.5	0.31
Sharpe ratio	0.33	0.27

et al. (2007) find that ambiguity results in out-of-sample Sharpe ratios greater than that of mean-variance portfolios. This is partly because parameter uncertainty and aversion to this uncertainty reduces extreme positions. Ambiguity aversion dampens the well known result that mean-variance efficient portfolios perform poorly out-of-sample by in effect inducing endogenously short-sales constraints. The reason for this qualitatively different result is that I assume that stocks are perceived to differ in ambiguity, thereby resulting in a concentration in the non-ambiguous stock. This underdiversification leads in turn to a lower Sharpe ratio.

I find that ambiguity aversion provides a potential explanation for underdiversification, a fact established in a vast body of empirical evidence. Furthermore, differences in levels of ambiguity are particularly likely to be present when comparing domestic stocks or domestic stock indices to foreign stocks. Hence this result also presents a (partial) rationale for the home bias present in stock investments (thereby providing a theoretical foundation for the empirical findings of a relation between ambiguity aversion and home bias in Dimmock et al. (2015a)).

Calvet et al. (2007) find that less financially sophisticated households tend to stay out of the stock market, and when investing they hold lower risky shares and obtain lower Sharpe ratios. These findings are consistent with the life-cycle model with ambiguity and learning, since lower financial sophistication is essentially similar to facing more ambiguity about stock investments. Less sophisticated household are more unsure and pessimistic of their own investment abilities, which combined with ambiguity aversion induces household to stay out of the market. Furthermore, when investing, less sophisticated, more ambiguous households are more likely to invest inefficiently. Calvet et al. (2007) show that taking into account that less financially literate household would make suboptimal investment choices results in a reduction of welfare costs of non-participation by almost one-half.

C. Effect of ambiguity aversion on savings

The optimal mean wealth levels are plotted in Figure 3. Agents who face ambiguity about the equity risk premium have lower amounts of wealth compared to individuals not facing ambiguity. For instance, at age 65 the mean wealth level when facing ambiguity is $225,000, compared to $250,000 when exposed to only risk. Note that the sizes of these wealth levels are comparable to the findings of Cocco et al. (2005).

These lower wealth levels for ambiguity-averse agents do not necessarily imply lower savings out of income and wealth. Several factors are resulting in differential wealth levels. First, these lower wealth levels are (partly) an automatic result of ambiguity-averse agents investing less in equity and thus having less wealth (savings plus investment return) accumulated. An additional rationale for lower wealth levels could be that individuals have fewer incentives to save, because they make investment decisions based on the worst case equity premium, which does not generate sufficient investment income on their savings. Going in the opposite direction are ambiguity-averse agents saving more to compensate for their lower wealth levels due to lower investment returns.

Figure 3: Optimal wealth levels

This figure shows the average optimal wealth for (1) agents ambiguous about the equity risk premium, averse to this ambiguity, and who learn about the equity premium and (2) agents who are not ambiguous about the equity premium.

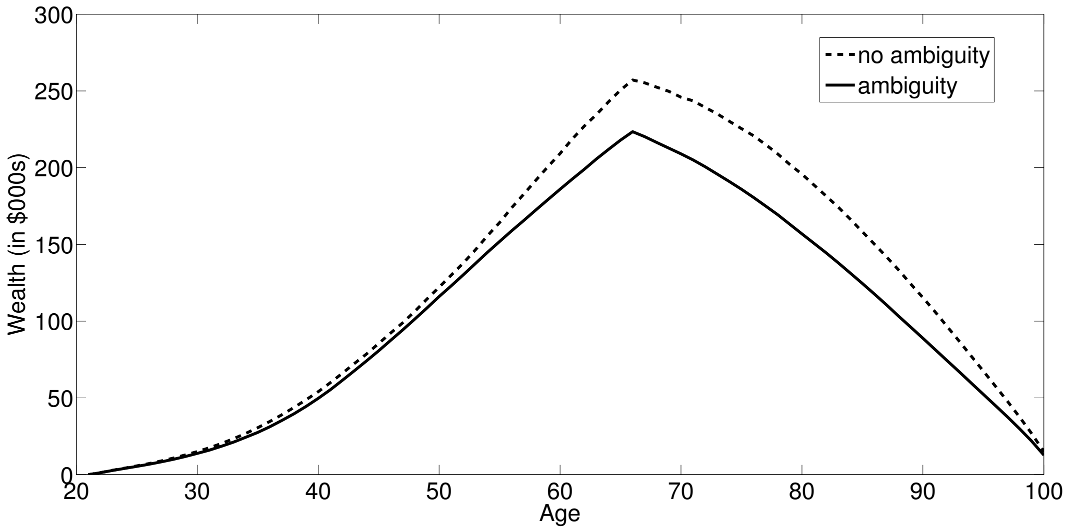
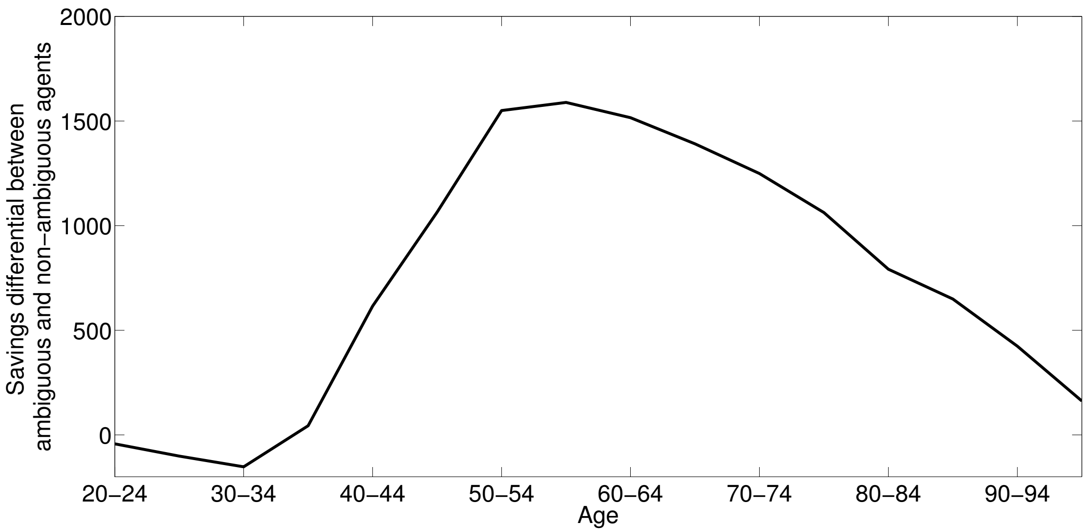


Figure 4 shows the difference in savings levels between individuals facing ambiguity and those not facing ambiguity with similar wealth and income levels. Ambiguity-averse individuals save substantially more than non–ambiguity-averse individuals. If the lower wealth levels of ambiguity-averse agents are merely a reflection of lower investment returns, then the savings out of income and wealth should be the same for both and the difference in Figure 4 should be zero. However, the savings out of income and wealth are higher for ambiguity-averse agents. Thus, keeping up wealth levels by saving extra is quantitatively more important than lower savings rates induced by the relatively less attractive investment opportunities perceived by ambiguity-averse investors.[12]

[12]Sutter et al. (2013) find that more ambiguity-averse adolescents save less; however, these results are insignificant.

Figure 4: Difference in savings between individuals facing ambiguity and not facing ambiguity
This figure shows the difference in the average savings out of income and wealth between agents who are ambiguity averse and those who are not. For a given wealth and income level (median wealth and income at each age for the non-ambiguous agent), savings are calculated for both types of agents, ambiguity averse and not ambiguity averse. The optimal savings for the ambiguity-averse agent minus the optimal savings for the non–ambiguity-averse agent are displayed.



IV. Comparing optimal allocations to the empirical evidence

In this section I compare the predictions from the life-cycle model with data from the Survey of Consumer Finances in 2010.

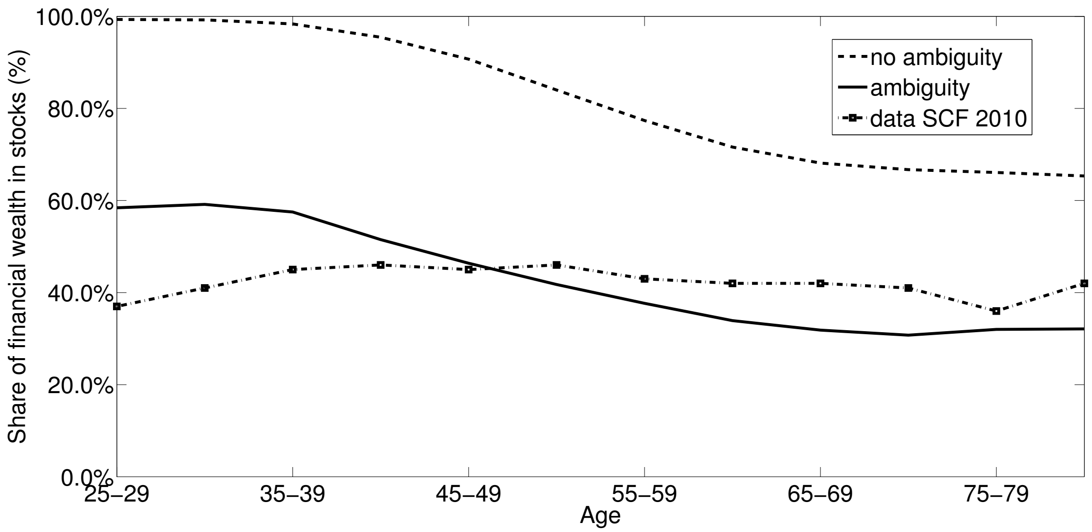
A. Fraction of financial wealth allocated to stocks

Figure 5 compares the optimal fraction allocated to stocks with the empirical levels in 2010. The fraction of financial wealth allocated to stocks for agents facing ambiguity is much closer to the empirical levels. The model without ambiguity predicts an optimal average fraction over the life cycle of about 85%, versus 45% when ambiguity is taken into account. The empirical average is slightly less than 45%.

The previous graph focuses on the means of the conditional allocation to stocks, not examining

Figure 5: Fraction of financial wealth allocated to stocks, conditional on participation: Empirical and optimal allocations

This figure shows the empirical fraction of financial wealth allocated to stocks, conditional on participation, and the optimal fraction invested in the stock market for (1) agents ambiguous about the equity risk premium, averse to this ambiguity, and who learn about the equity premium and (2) agents who are not ambiguous about the equity premium. The data are from the 2010 Survey of Consumer Finances.



other moments. Table 2 displays the stock allocations predicted by the model and the empirical estimates for different percentiles. For all ages grouped together (ages 25–84), the median matches well, with 40% of financial wealth allocated to stocks in the data compared to 39% according to the model with ambiguity. Furthermore, for instance, the 10th percentile is 7% in the data, 10% according to the model that includes ambiguity, and 67% for the model without ambiguity. When splitting the fraction invested in stocks into different age groups, the fit to the data is similar. In line with the data, the model with ambiguity generates a large degree of heterogeneity in conditional fractions allocated to stocks, in contrast to the model without ambiguity.

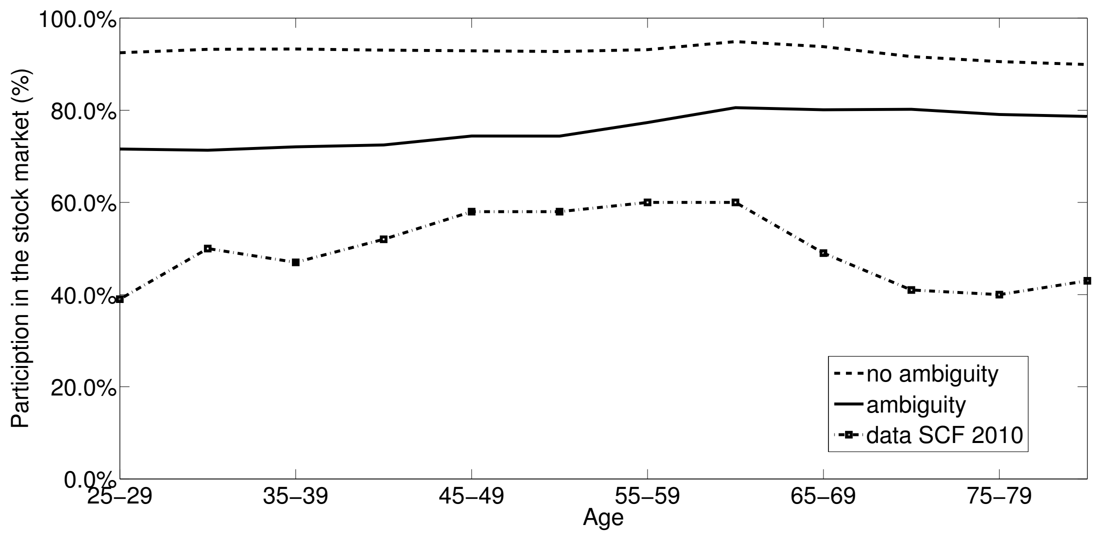
Table 2: Percentiles of optimal and empirical fractions of financial wealth allocated to stocks
The optimal fraction is calculated using the simulation results of the benchmark life-cycle model for (1) agents ambiguous about the equity risk premium, averse to this ambiguity, and who learn about the equity premium and (2) agents who are not ambiguous about the equity premium. Both the optimal and empirical fractions are conditional on stock market participation. The data are from the 2010 Survey of Consumer Finances.

Data	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Ages 25–84	7	19	40	65	86
Ages 25–34	6	15	35	62	84
Ages 35–44	9	22	42	69	89
Ages 45–54	9	21	43	67	89
Ages 55–64	6	19	41	62	84
Ages 65–74	7	18	37	65	81
Ages 75–84	5	12	36	58	80
Model with ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Ages 25–84	10	21	39	62	84
Ages 25–34	14	32	59	93	100
Ages 35–44	12	28	52	84	100
Ages 45–54	9	20	38	65	98
Ages 55–64	7	16	29	49	81
Ages 65–74	8	16	27	42	62
Ages 75–84	8	17	28	42	62
Model without ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Age 25–84	67	73	81	94	100
Ages 25–34	100	100	100	100	100
Ages 35–44	90	99	100	100	100
Ages 45–54	62	76	96	100	100
Ages 55–64	51	57	71	98	100
Ages 65–74	48	53	60	84	100
Ages 75–84	47	51	58	80	100

B. Stock market participation

Figure 6 plots the optimal stock market participation and empirically observed participation levels. Comparing the empirical participation levels to the optimal participation levels when taking into account ambiguity, I find a close match. In 2010, averaged over the entire life cycle, about 50% of the people invest in the stock market and the model predicts about 75%. To compare, Gomes and Michaelides (2005) find optimal allocation levels of almost 100% at young ages, while the model with ambiguity predicts optimal levels of a bit more than 70%. The fit would be even better when assuming a less conservative level for the ambiguity about the equity premium. In the baseline case the standard deviation of the belief about the equity premium at birth is 2%, which implies that due to learning the standard deviation of the belief is 1.76% at age 20. The results for a level of ambiguity of 3% at birth is presented in Figure 10. Note that it is not insightful to present the percentiles for the participation levels, since this is a binary variable and all the information is already contained in Figure 6.

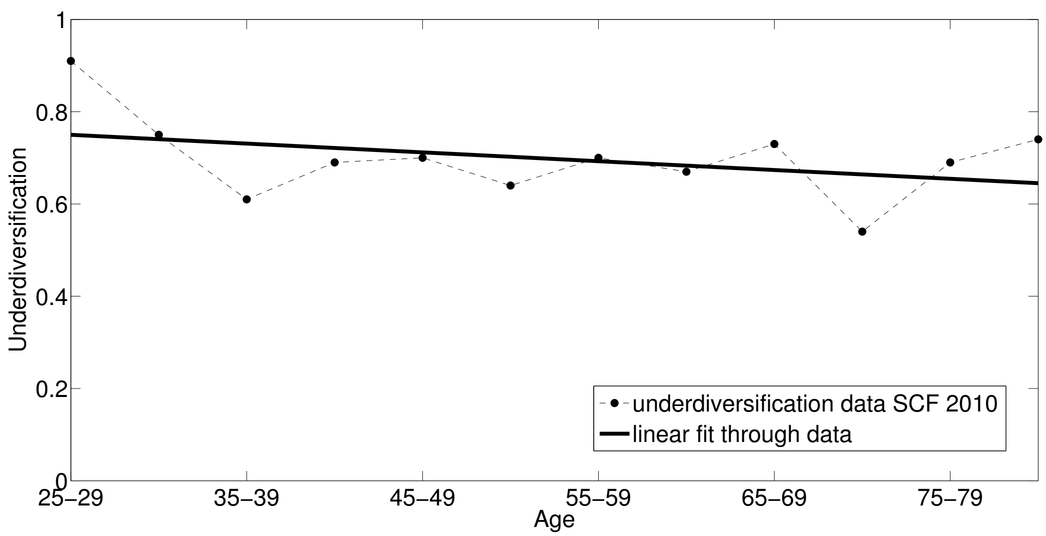
Figure 6: Stock market participation: Empirical and optimal allocations
I display the empirical participation levels and optimal participation levels in the stock market for (1) agents ambiguous about the equity risk premium, averse to this ambiguity, and who learn about the equity premium and (2) agents who are not ambiguous about the equity premium. In case an agent has a near-zero financial wealth level (below \$100), optimal participation is assumed to be zero. The data are from the 2010 Survey of Consumer Finances.



C. Underdiversification over the life cycle

The model predicts that ambiguity-averse individuals hold more underdiversified portfolios, which is consistent with the survey results in Dimmock et al. (2015a). A direct comparison of the ambiguity measures in the model to the empirical evidence is not possible due to the necessary abstraction of modelling two assets with varying degrees of ambiguity. However, in this section I compare the theoretical life-cycle pattern of underdiversification to the data. Figure 7 shows the empirical levels of underdiversification over the life cycle. The measure for underdiversification is the fraction of the total stock portfolio held in individual stocks, which Calvet et al. (2007) show is a reasonable proxy for portfolio underdiversification as most individuals hold only a couple of different individual stocks. Similar to the model, the empirical evidence indicates that underdiversification decreases (slightly) with age. This pattern is clear especially when focussing on the solid line, which is a first-order polynomial fitted through the data points.

Figure 7: Empirical levels of underdiversification over the life cycle
Underdiversification is measured as the investment in individual stocks as a fraction of total stock investments. I use data from the Survey of Consumer Finances in 2010. More details are in Appendix C.



D. Wealth-to-income ratios

In this section, I compare the wealth-to-income ratios predicted by the model with the data. To facilitate comparison I use the same definition of the wealth-to-income ratio and similar age groups as in Gomes and Michaelides (2005), details are in Appendix C. Table 3 shows the 10th, 25th, 50th, 75th, and 90th percentiles for three age groups: buffer stock savers (20-34), retirement savers (35-

64), and retirees (over 65). The model predicts large heterogeneity in wealth accumulation, in all age groups, as is found in the data. The model with ambiguity and without ambiguity generate comparable wealth-to-income ratios. Low wealth-to-income ratios for the poorer households (10th percentile) are matched well with the data. Furthermore, the wealth-to-income ratio is matched especially well for buffer stock savers (age 20-34). Overall, the distribution of wealth-to-income ratios generated by the model are comparable to the data. A similar good fit is documented in Gomes and Michaelides (2005).[13]

Table 3: Percentiles of optimal and empirical wealth-to-income ratios
The wealth-to-income ratios in 2010 are calculated using the Survey of Consumer Finances. More details are in Appendix C.

Data	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Ages 20–34	0.00	0.03	0.23	0.84	1.81
Ages 35–64	0.00	0.27	1.48	4.00	8.24
Ages 65–end	0.08	1.81	5.66	12.87	26.91
Model with ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Ages 20–34	0.00	0.14	0.42	0.95	1.75
Ages 35–64	0.13	1.05	3.47	7.18	11.68
Ages 65–end	0.14	3.21	8.23	13.61	20.13
Model without ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Ages 20–34	0.01	0.15	0.45	1.04	1.92
Ages 35–64	0.16	1.16	3.66	8.11	14.01
Ages 65–end	0.18	3.77	9.87	17.40	26.80

V. Sensitivity analysis

This section tests the importance of several features of the model. I explore whether higher risk aversion can substitute for ambiguity aversion and thereby change the optimal stock allocation to be in line with the data. Furthermore, the impact of the age at which the agent starts learning

[13] I also examined savings rates, which are notoriously hard to estimate empirically. The predicted savings rates by the model with ambiguity fall within the range of empirical estimates in Dynan et al. (2004), where they use different datasets and methods to calculate savings rates. Results available from the author upon request.

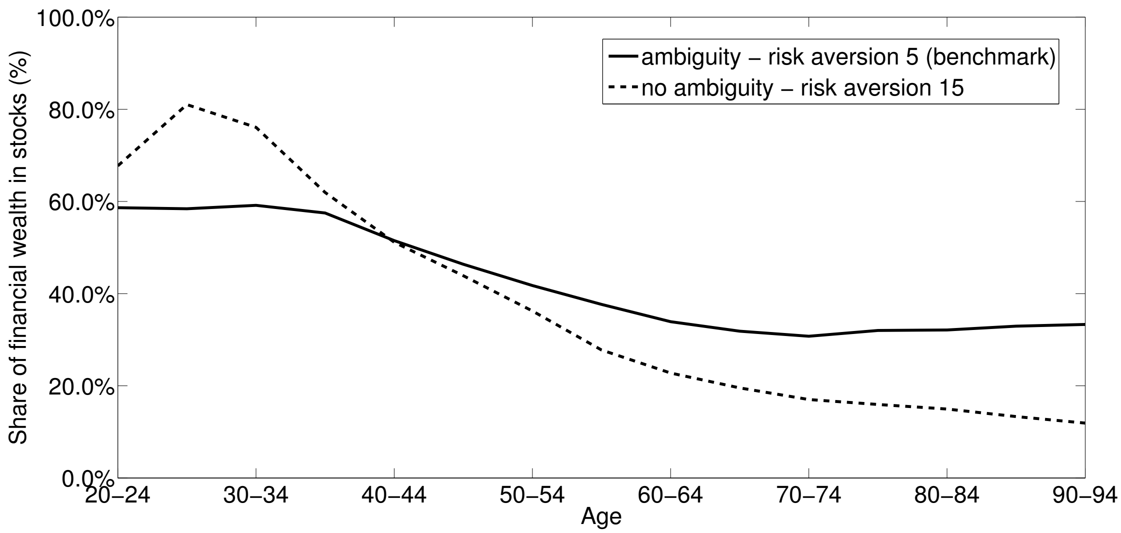
and the influence of the initial level of ambiguity about the equity premium are shown. Finally, an alternative ambiguity model, α-maxmin preferences, is examined.

A. Can higher risk aversion substitute for ambiguity aversion?

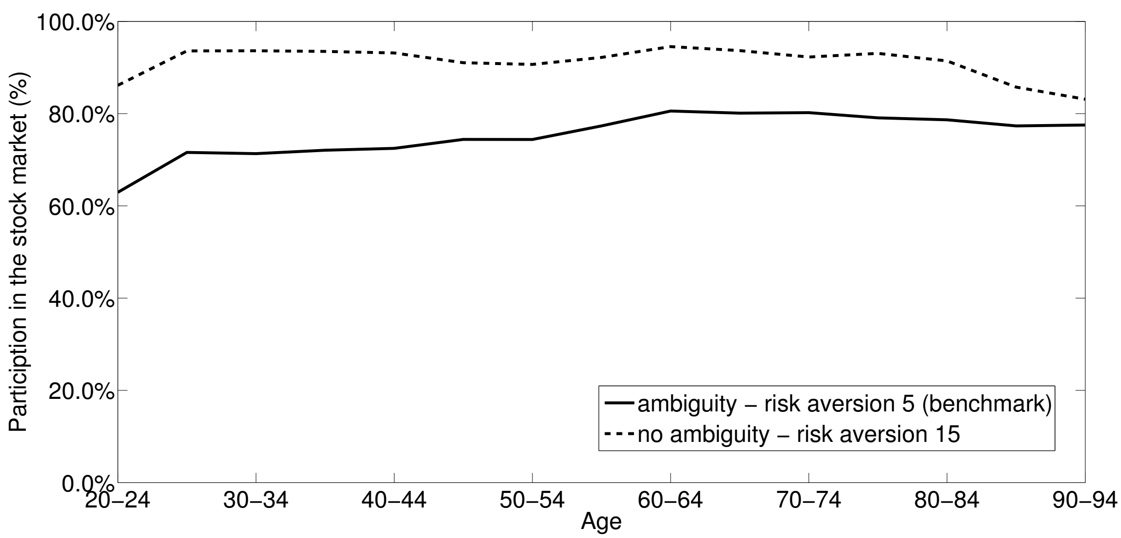
This paper shows that ambiguity about the equity risk premium can help solve the participation puzzle and explain the low fraction of financial wealth allocated to stocks over the life cycle. In this section I show that similar findings cannot be obtained by assuming higher risk aversion. The results are presented in Figure 8. Arguably, when agents have a risk aversion of 15 and are not ambiguity averse, the optimal fraction matches well the empirically observed fractions allocated to stocks. However, in Figure 8b is shown that high risk aversion has almost no influence on optimal participation levels, close to 100% participation is predicted. Higher risk aversion actually increases participation, since it increases precautionary savings. The key difference is that while ambiguity aversion depresses participation levels, risk aversion cannot. Risk averse agents will always optimally invest a positive fraction of their financial wealth in stocks, when the equity premium is positive. Ambiguity-averse agents, if the worst case belief is zero or negative, do not participate in the stock market, whereas if the worst case equity risk premium is positive, the agent participates. Hence risk aversion does not act as a substitute for ambiguity aversion and I do not obtain the same results with higher risk aversion compared to those obtained when including ambiguity aversion.

Figure 8: Stock allocations: Can risk aversion substitute for ambiguity aversion?
These figures show the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) agents ambiguous about the equity risk premium, averse to this ambiguity, and who learn about the equity premium and (2) agents who are highly risk averse (gamma=15) and not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation



B. Impact of starting learning at a later age

In the benchmark model, I assume that people incorporate in their beliefs returns realized from the year of birth onward, equally weighting each stock market realization. This section explores the impact of a different approach for updating beliefs about the equity premium. Specifically, I assume that agents learn from age 20 onward instead of from birth and the results are plotted in Figure 9. Both the optimal participation levels and the fraction of financial wealth allocated to stocks are lower if agents update beliefs from age 20 onward (compare to Figure 1). The reason is that the level of ambiguity, the standard deviation of beliefs about the equity premium, is higher when updating starts at a later age. When individuals are assumed to incorporate stock market realizations from age 20 onward, they will have seen only one stock market realization at age 21, compared to 21 realizations when all realizations after birth are incorporated. Hence, assuming learning starts from age 20 strengthens the main results in this paper; the impact of ambiguity about the equity premium on optimal stock allocations is larger. Also, when comparing to the optimal allocations in Figures 5 and 6, the fit improves when assuming updating from age 20. Therefore, the benchmark updating from birth onward presents a conservative assessment of the impact of ambiguity on portfolio choice.

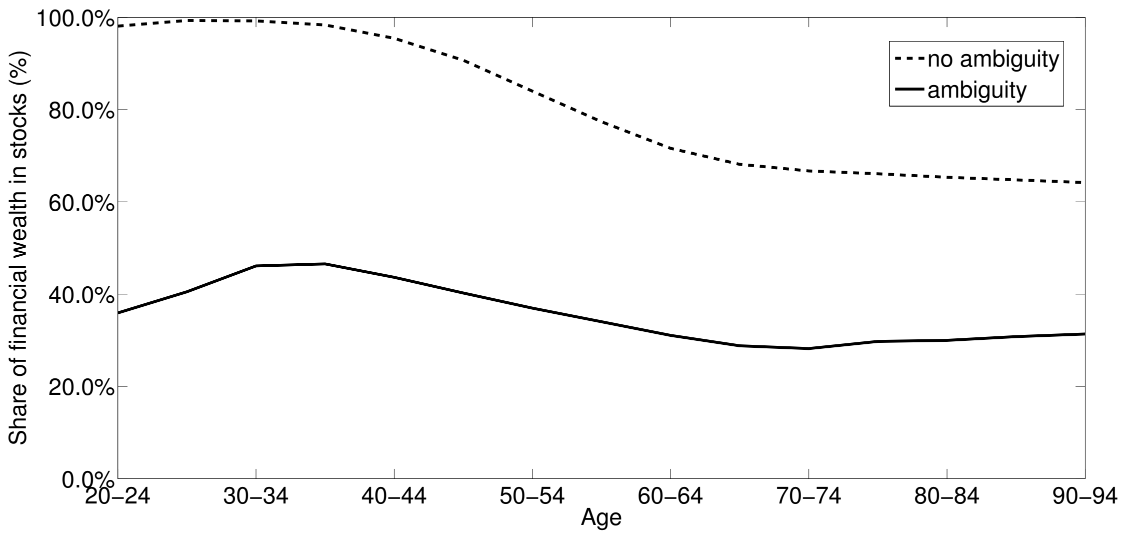
C. Impact of initial ambiguity about the equity risk premium

The initial level of ambiguity, that is, the standard deviation of the belief about the equity risk premium, is set at 2% in the benchmark case. Intuitively, a standard deviation of 2% seems reasonable, since this ensures that the 95% confidence interval of the equity risk premium that the agent believes is possible is between 0% and 8% at birth (and on average between 0.6% and 7.4% at age

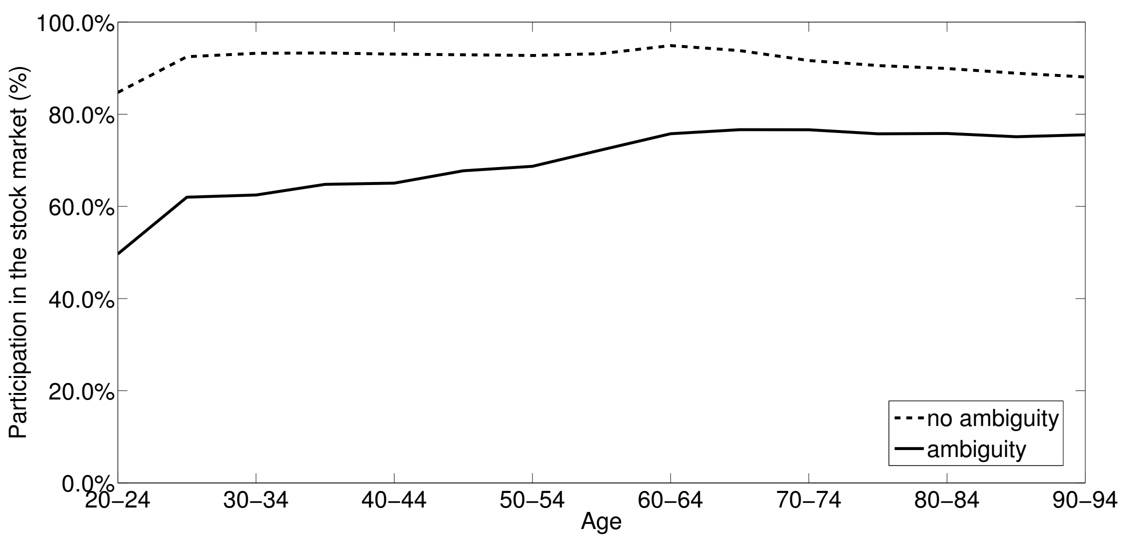
Figure 9: Stock allocations: Learning from age 20 onward

These figures show the impact of learning from age 20 onward instead of from the year of birth. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) agents ambiguous about the equity risk premium, averse to this ambiguity, and who learn about the equity premium and (2) agents who are not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

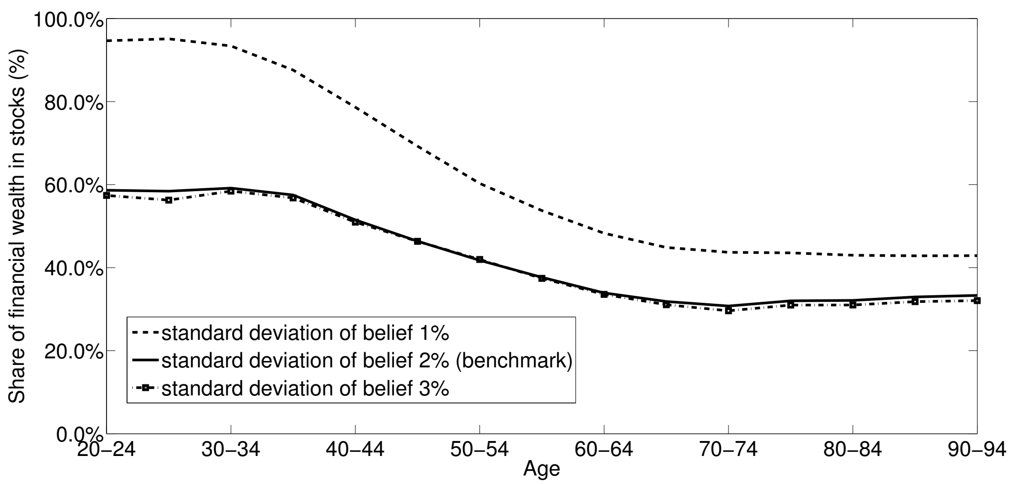


20). Compelling evidence that this is not overstating the degree of ambiguity but is rather conservative can be derived from the financial literacy literature. When answering questions to establish financial literacy levels, Rooij van et al. (2011) find that 22% of survey respondents answer that they do not know whether "considering a long time period, stocks, bonds, or savings accounts give the highest return". Furthermore, 30% gives the wrong answer and less than half gives the correct answer. This at least indicates that a large fraction of agents being ambiguous about the equity premium and, generally, about financial market parameters is a valid assumption. Furthermore, even assuming that agents look up all previous stock market returns, the confidence interval about the equity premium would still be large. However, since I have no means to determine the range of equity risk premium that agents deem possible, this section examines the influence of the initial ambiguity level on the results. These results can also be viewed in light of agents having different levels of ambiguity and how this influences the optimal fraction allocated to stocks and optimal participation levels.

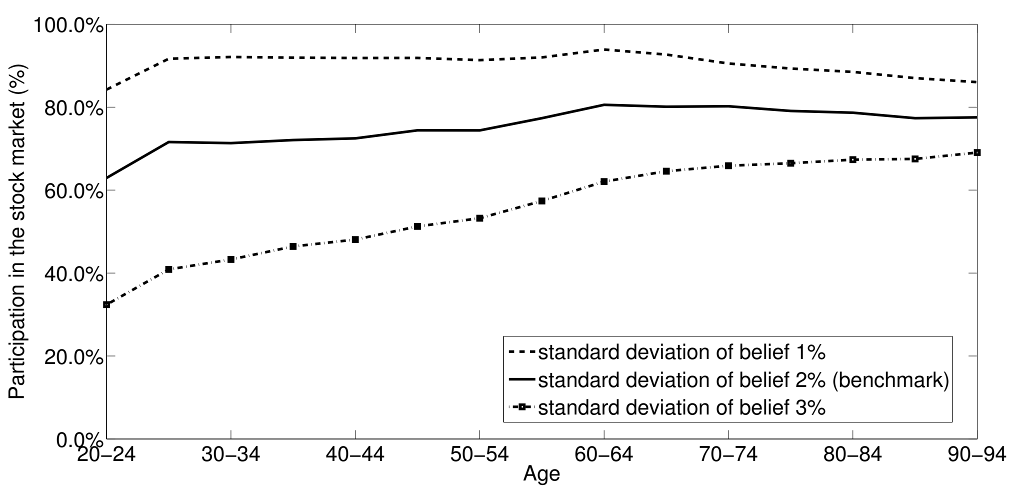
Figure 10 displays conditional allocations to equity and participation in the stock market for three different levels of initial ambiguity. The optimal conditional fraction allocated to stocks is approximately the same if the *initial* standard deviation of beliefs is 3% or 2%. The reason is that agents only participate in the stock market if, after updating, they have a positive worst case belief. Then, conditional on participation, the average belief about the equity premium is the same for agents starting out with a standard deviation of beliefs of 2% and 3%, and thus the average conditional fraction is the same. The fraction allocated to stocks if agents have a standard deviation of beliefs of 1% is much higher at young ages, because the average worst case belief, conditional on having a positive worst case belief, is higher. Participation levels differ substantially, since the agent with a standard deviation of 3% needs a much larger positive average realized return to have a worst case belief higher than 0% and participate, compared to an agent with a 2% standard deviation.

Figure 10: Stock allocations: Three different initial levels of ambiguity
These figures show the impact of the initial level of ambiguity. A standard deviation of the belief about the equity premium of 1%, 2%, or 3% is assumed. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), the optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation



D. Alternative ambiguity model: *α-maxmin preferences*

The previous sections show that ambiguity aversion has a large effect on optimal portfolio choices for agents with maxmin preferences. However, there is no consensus on whether agents exhibit maxmin preferences or other specifications of ambiguity preferences. A potential critique can be that maxmin preferences are rather extreme as the agent considers only the belief generating the lowest expected utility and maxmin does not allow a separation of ambiguity and ambiguity aversion. Hence this section examines the influence of ambiguity aversion when agents behave according to α-maxmin preferences (Chateauneuf et al. (2007) and Ghirardato et al. (2004)). These ambiguity-averse preferences allow a less than 100% weight on the belief generating the lowest expected utility and it allows a separation between beliefs and attitudes. The investor solves the following Bellman equation at time $t \neq T$:

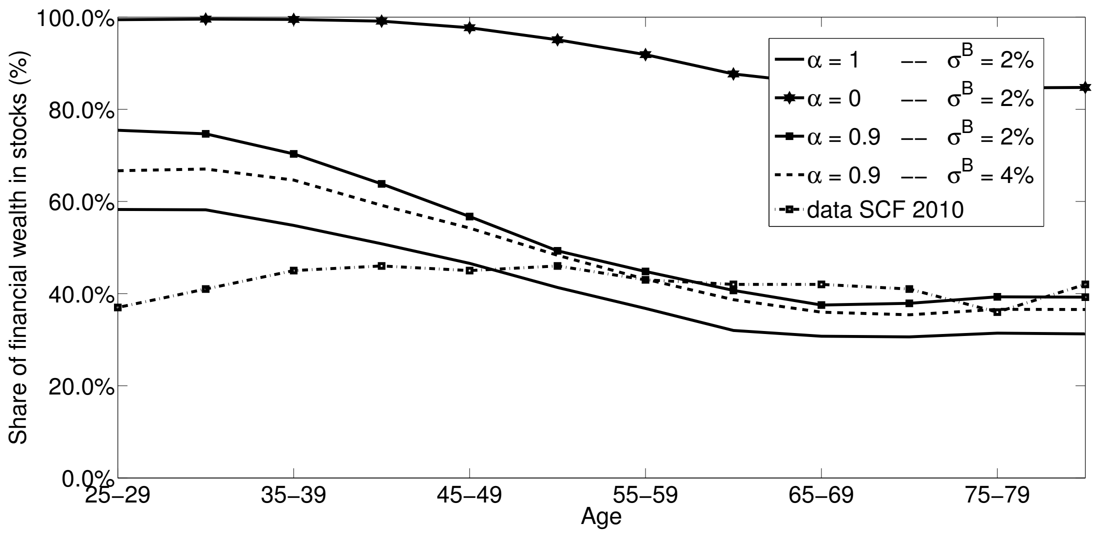
$$\begin{aligned} V_t(W_t, Y_t, \lambda_t^B) &= \max_{w_t, C_t} \alpha[\min_{\lambda \in \Lambda_t}\{u(C_t) + \beta p_{t+1}\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B))\right]\}] + \\ &\quad (1-\alpha)[\max_{\lambda \in \Lambda_t}\{u(C_t) + \beta p_{t+1}\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B))\right]\}], \end{aligned} \tag{12}$$

where α represents an ambiguity-aversion parameter. In case α is equal to one, all the weight is put on the worst case, which is the maxmin model. If α is zero, all the weight is put on the best case, which means agents are ambiguity loving. As of yet only a few papers try to estimate this parameter and arrive at varying numbers, for instance Dimmock et al. (2015b) find 0.56. The α-maxmin utility function can be interpreted as attaching a weight α to the belief generating the minimum expected utility and a weight $(1-\alpha)$ to the belief generating the maximum expected utility. Details on the numerical solution method are given in Appendix D.

Figure 11: Stock allocations: Alternative ambiguity aversion model - α-maxmin

These figures show the impact of an alternative model for ambiguity aversion, that is, α-maxmin preferences. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) $\alpha = 1$ & $\sigma^B = 0.02\%$, (2) $\alpha = 0$ & $\sigma^B = 0.02\%$, (3) $\alpha = 0.9$ & $\sigma^B = 0.02\%$, and (4) $\alpha = 0.9$ & $\sigma^B = 0.04\%$. Furthermore, the empirical fraction of financial wealth allocated to stocks and participation in the stock market is displayed. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

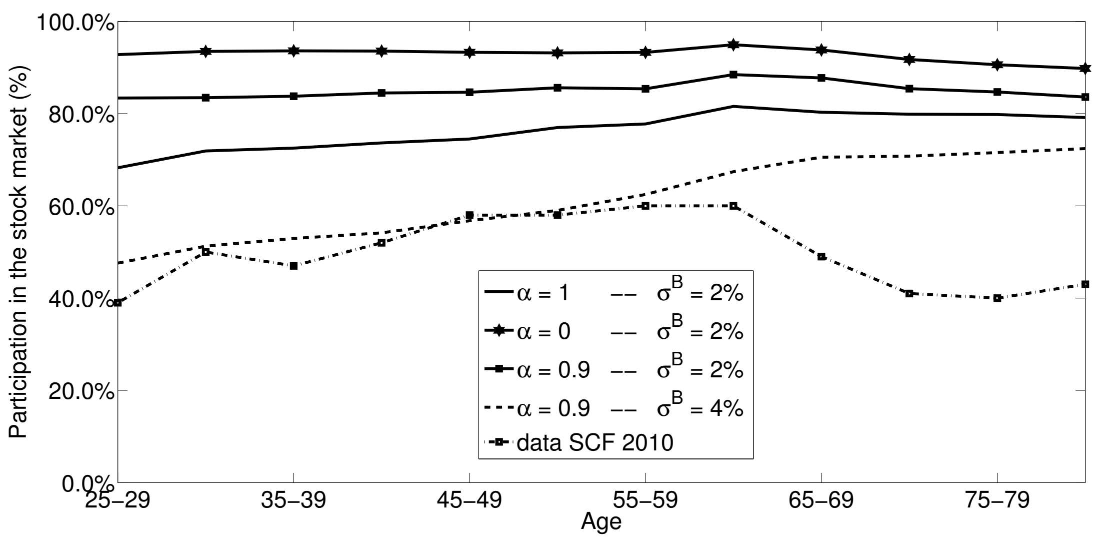


Figure 11a shows the optimal fraction of financial wealth allocated to stocks, conditional on participation, for several levels of the ambiguity aversion parameter α and several levels of the standard deviation of the beliefs at birth, σ^B. In case α is equal to one, all the weight is put on the worst case, which results in optimal allocations identical to the benchmark maxmin preferences (similar to Figures 5 and 6).[14] In the other case of α equals zero, all the weight is put on the highest equity premium belief, which implies ambiguity seeking behaviour. This results in a fraction allocated to stocks that is on average more than 90%, which is higher than in the case of no ambiguity (dashed line in Figure 5). If $\alpha = 0.9$ and the standard deviation of the beliefs at birth is as in the baseline case, the fit to the data is slightly lower compared to the baseline maxmin preference specification. Assuming a higher σ^B of 4%, combined with an ambiguity aversion parameter of 0.9, provides a good fit to the data. Similar results follow from the participation in the stock market. Overall, I find the best fit to the data for high levels of the ambiguity aversion parameter α.

VI. Conclusion

This paper develops a realistically calibrated life-cycle model with ambiguity aversion and learning to explore the impact of ambiguity about the equity risk premium on optimal portfolio allocations. Taking into account ambiguity about the equity premium reduces optimal participation levels, on average, by 20% and the optimal conditional fraction of financial wealth allocated

[14]An alternative preference specification would be smooth recursive preferences (Klibanoff et al. (2005)), which is numerically challenging to solve for high levels of the ambiguity aversion parameter. When using smooth recursive preferences, ambiguity aversion does not reduce to choosing the lowest admissible equity premium. Note that smooth ambiguity preferences with infinite ambiguity aversion is equivalent to maxmin preferences.

to stocks by 40%. When focussing on allocations within the stock portfolio, I find that ambiguity aversion leads to underdiversification, home bias, and lower Sharpe ratios. Furthermore, wealth levels over the life cycle are reduced substantially while savings out of income and wealth are increased. I compare the model predictions with data from the Survey of Consumer Finances. Two important empirical facts are matched: the low participation levels in the stock market over the life cycle and the low fraction of financial wealth allocated to equity, conditional on participation. Furthermore, consistent with empirical evidence using the SCF, underdiversification decreases with age.

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A. Numerical method for model with maxmin preferences

Investors preferences are described by maxmin expected utility, which, in effect, means that agents maximize their utility with respect to the worst case belief. Agents are uncertain about the equity risk premium. I solve the following Bellman equation:

$$V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t, C_t} \min_{\lambda \in \Lambda_t} \{u(C_t) + \beta p_{t+1} E_t^{\lambda} \left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B) \right] \}, \text{ with} \tag{A.1}$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \tag{A.2}$$

subject to (3) to (6). As described in Section II.A, I restrict the domain of beliefs about the equity risk premium to lie in $[\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$. This is necessary otherwise the worst case belief would be unbounded, since the beliefs are normally distributed.

In each period the optimal asset weights are given by the first-order condition:

$$E_t^{\lambda_t^{min}}[C_{t+1}^{*-\gamma}(R_{t+1} - R^f)] = 0, \tag{A.3}$$

where λ_t^{min} is the lowest equity premium belief in Λ_t and C_{t+1}^* denotes the optimal real consumption level. Optimal consumption follows from

$$C_t^{*-\gamma} = \beta p_{t+1} E_t^{\lambda_t^{min}} \left[C_{t+1}^{*-\gamma} R_{t+1}^{P*} \right], \tag{A.4}$$

where R^{P*}_{t+1} is given by (6) evaluated at the optimum.

I regress the realizations of the Euler condition on a polynomial expansion in the state variables to obtain an approximation of the conditional expectation in the Euler condition

$$E_t^{\lambda_t^{min}}\left[C_{t+1}^{*-\gamma}(R_{t+1}-R^f)\right] \simeq \rho(w_t, A_t)' h(Y_t, \lambda_t^B), \tag{A.5}$$

where $A_t = W_t + Y_t - C_t$ is defined as the after consumption wealth. In addition, I employ a further approximation, introduced by Koijen et al. (2010), who show that the regression coefficients ρ are smooth functions of the asset weights and, consequently, I approximate the regression coefficients ρ by projecting them further on a polynomial expansion in the asset weights:

$$\rho(w_t, A_t) \simeq \Psi(A_t) g(w_t). \tag{A.6}$$

The Euler condition must be set to zero to find the optimal asset weights:

$$g(w_t)' \Psi(A_t)' h(Y_t, \lambda_t^B) = 0. \tag{A.7}$$

Similarly, I approximate the Euler condition for optimal consumption, equation (A.4), via regressing the realization of the Euler conditions on a polynomial expansion in the state variables. This results in optimal policies for every simulation path. I simulate 5000 paths and repeat this 20 times. Results are obtained by using the total of 100,000 simulation paths. A more detailed explanation of the numerical approach is described in Koijen et al. (2010).

B. Model with two assets and maxmin preferences

I model two risky assets to explore the impact of ambiguity aversion on underdiversification. Both assets are risky, however one asset is ambiguous, while the second asset is not ambiguous. The first asset, ambiguous and risky, is identical to the asset in our baseline model described in Section II. The second asset, risky but not ambiguous, has the same true equity premium and annual volatility as the ambiguous asset. The only difference is that the agent takes these parameters as a given, and does not update his beliefs according to stock return realizations.

B.1. Summary of the life-cycle problem with two assets

The investor solves the following Bellman equation at time $t \neq T$:

$$V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t^A, w_t^{NA}, C_t} \min_{\lambda \in \Lambda_t} \left\{ u(C_t) + \beta p_{t+1} \mathbb{E}_t^{\lambda} \left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)) \right] \right\}, \tag{B.1}$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \tag{B.2}$$

where w_t^{NA} is the fraction of wealth allocated to the non-ambiguous asset and w_t^A is the fraction of wealth allocated to the ambiguous asset.

The individual faces the constraint (4) and borrowing and short-sales constraints

$$w_t^{NA} \geq 0 \text{ and } w_t^{NA} \leq 1 \tag{B.3}$$

$$w_t^A \geq 0 \text{ and } w_t^A \leq 1 \tag{B.4}$$

$$w_t^A + w_t^{NA} \leq 1. \tag{B.5}$$

The intertemporal budget constraint equals

$$W_{t+1} = (W_t - C_t + Y_t)R_{t+1}^P, \tag{B.6}$$

where R_{t+1}^P denotes the portfolio return

$$R_{t+1}^P = 1 + R^f + (R_{t+1}^{NA} - R^f)w_t^{NA} + (R_{t+1}^A - R^f)w_t^A \tag{B.7}$$

and R_{t+1}^{NA} and R_{t+1}^A denote the stock return between time t and $t+1$ on the non-ambiguous and ambiguous stock, respectively.

B.2. First-order conditions: two risky assets

In each period the optimal asset weights in the non-ambiguous and ambiguous assets are given by the first-order conditions:

$$E_t^{\lambda_t^{min}}[C_{t+1}^{*-\gamma}(R_{t+1}^{NA} - R^f)] = 0 \tag{B.8}$$

$$E_t^{\lambda_t^{min}}[C_{t+1}^{*-\gamma}(R_{t+1}^{A} - R^f)] = 0, \tag{B.9}$$

where λ_t^{min} is the lowest equity premium belief in Λ_t and C_{t+1}^* denotes the optimal real consumption level. Optimal consumption follows from

$$C_t^{*-\gamma} = \beta p_{t+1} E_t^{\lambda_t^{min}}\left[C_{t+1}^{*-\gamma} R_{t+1}^{P*}\right], \tag{B.10}$$

where R_{t+1}^{P*} is given by (B.7) evaluated at the optimum.

C. Survey of Consumer Finances and stock allocations

The Survey of Consumer Finances is a triennial survey on the financial assets of the household. It provides information on assets on the balance sheet, pensions, income, and demographics of the household. Participation in the survey in strictly voluntary and about 4500 families are interviewed. It is a repeated cross section and only the years 1983 to 1989 are part of the panel study. The median length of an interview is about 75 minutes, but an interview with a family with complex finances can take up to several hours. High-income households are over-sampled to measure asset holdings more accurately, since wealth in the United States is highly concentrated among a relatively small number of households. About two-thirds of the sample, 3000 households, is drawn from a national area probability sample that represents the entire population. The remaining one-third, 1500 households, is drawn from tax records to obtain the list of high-income households. Weights are used to account for both non-response and the difference between the initial sample design and the actual distribution of population characteristics. In the case of missing data, multiple imputation is used to solve this problem.

Financial wealth is the sum of liquid assets (checking, savings, money market, and call accounts); certificates of deposit; directly held mutual funds; stocks; bonds; quasi-liquid retirement accounts that consists of IRAs/Keoghs, thrift accounts, and future pensions; savings bonds; the cash value of whole life insurance; other managed assets (trusts, annuities, and managed investment accounts); and other financial assets (loans from the household to someone else, future proceeds, royalties, futures, non-public stock, deferred compensation, oil/gas/mineral investment).

The part of financial assets invested in stocks consists of directly held stock, stock mutual funds, and retirement assets invested in stocks. I follow the Survey of Consumer Finances in calculating the financial wealth and stock investments. Stock investments includes all directly held stock, all stock mutual funds, half of the value of combination mutual funds, and a fraction of the value of IRAs/Keoghs that is invested in stocks. Similarly, the fraction of the value of other managed assets invested in stocks is added and the part of the value of the thrift account that is allocated to stocks.

The fraction of agents participating in the stock market is determined by calculating which weighted fraction in the total sample has a stock investment larger than zero. Furthermore, the conditional allocation to equity is the fraction allocated to stocks, conditional on participation in the stock market. The measure for underdiversification is the fraction of the stock portfolio in directly held stocks. Note that I use weights to calculate the participation rate, the conditional allocation to stocks, and underdiversification to adjust for non-response and the non-equal probability design of the survey.

The wealth-to-income ratio is constructed as in Gomes and Michaelides (2005) and matches the definition of income in the life-cycle model as close as possible. Labor income is the sum of the income from wages and salaries, unemployment or worker's compensation, Social Security or other pensions, annuities, or other disability or retirement programs. Wealth is defined as the sum of financial wealth (previously defined) and net home equity.

D. Asset allocation with alternative ambiguity preferences

D.1. Summary of the life-cycle problem with α-maxmin preferences

$$V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t, C_t} \alpha[\min_{\lambda \in \Lambda_t} \{u(C_t) + \beta p_{t+1} \mathbb{E}_t^{\lambda} \left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)) \right] \}] +$$

$$(1-\alpha)[\max_{\lambda \in \Lambda_t} \{u(C_t) + \beta p_{t+1} \mathbb{E}_t^{\lambda} \left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)) \right] \}], \tag{D.1}$$

subject to (5). The ambiguity aversion parameter is represented by α. The exogenous state variables are income, Y_t, and the mean of the belief about the equity premium λ_t^B. Wealth, W_t, is an endogenous state variable.

D.2. Beliefs about the equity risk premium

As described before, each agent has beliefs about the equity risk premium with mean λ_t^B and standard deviation σ_t^B. I limit the set of beliefs that the agent thinks are viable to be bounded by a 95% confidence interval. Hence the beliefs about the equity premium lie in the range of $[\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$. The belief that generates the minimum expected utility is $\lambda_t^B - 2\sigma_t^B$, and the belief that generates the maximum expected utility is $\lambda_t^B + 2\sigma_t^B$.

D.3. First-order conditions: α-maxmin preferences

In period T, the optimal policies are easily determined: Namely, the agent consumes the entire wealth level and no optimal investment strategy needs to be made. In all the other time periods, optimal decisions on consumption and investment are calculated by deriving the first-order conditions of the problem. The optimization problem is solved via dynamic programming and I proceed backward.

Consider the agent at time t, who has to choose C_t and w_t. The first-order conditions for this problem are:

$$\text{(D.2)} \qquad \frac{\partial V_t}{\partial w_t} = \alpha E_t^{\lambda_t^{min}}[C_{t+1}^{*-\gamma}(R_{t+1} - R^f)] + (1-\alpha)E_t^{\lambda_t^{max}}[C_{t+1}^{*-\gamma}(R_{t+1} - R^f)] = 0,$$

and

$$\text{(D.3)} \qquad C_t^{*-\gamma} = \alpha\beta p_{t+1} E_t^{\lambda_t^{min}}\left[C_{t+1}^{*-\gamma} R_{t+1}^{P*}\right] + (1-\alpha)\beta p_{t+1} E_t^{\lambda_t^{max}}\left[C_{t+1}^{*-\gamma} R_{t+1}^{P*}\right].$$

where R_{t+1}^{P*} is given by (6) evaluated at the optimum, and λ_t^{min} and λ_t^{max} are the lowest and highest equity premium belief in Λ_t, respectively.